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On wheel-free graphs

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Abstract: A wheel is a graph formed by a chordless cycle and a vertex that has at least three neighbors in the cycle. We prove that every 3-connected graph that does not contain a wheel as a subgraph is in fact minimally 3-connected. We prove that every graph that does not contain a wheel as a subgraph is 3-colorable.

Key-words: Truemper configuration, wheel, cycle through three vertices, coloring, minimally 3-connected graph

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Sur les graphes sans roues

Résumé : Une roue est un graph formé d'un cycle sans corde et d'un sommet ayant au moins trois voisins dans ce cycle. Nous prouvons que tout graphe 3-connexe dont aucun sous-graphe n'est une roue est minimalement 3-connexe. Nous montrons également que tout graphe dont aucun sous-graphe n'est une roue est 3-colorable.

Mots-clés : configuration de Truemper, roue, cycle par trois sommets, coloration, graphe minimalement 3-connexe

1 Introduction

A *wheel* is a graph formed by a chordless cycle C, called the *rim*, and a vertex v (not in $V(C)$), called the *center*, such that the center has at least three neighbors on the rim. So, the complete graph on four vertices is the smallest wheel. When convenient, we denote a wheel by (C, v). Wheels are one of the four *Truemper's configurations*, see [15], that play a role in several theorems on the structure of graphs and matroids. Let us see more precisely how wheels play some role in the description of the structure of several graph classes.

A *hole* in a graph is a chordless cycle on at least four vertices. The structure of a *Berge graph* G, that is a graph such that G and its complement do not contain odd holes, is studied in [3]. The results obtained there famously settled the Strong Perfect Graph Conjecture. The proof goes through several cases, and the last fifty pages of the proof deal with the case when G contains certain kinds of wheels. Consequently the structure of a Berge graph G is simpler when G does not contain these kinds of wheels. In addition, the structure of a graph G with no even holes is complex. A first decomposition theorem was given in [4] and a better one in [13]. In the later paper, very long arguments are devoted to situations when G contains certain kinds of wheels. This suggests that graphs that do not contain a wheel as an induced subgraph should have interesting structural properties. Understanding this structure might shed a new light on the works listed above. Since "understanding the structure" is a slightly fuzzy goal, we adress the following precise open questions.

Question. Is there a constant c such that every graph with no wheel as an induced subgraph is c-colorable?

Question. Is there a polynomial-time algorithm to decide whether an input graph contains a wheel as an induced subgraph?

As observed by Esperet and Stehlík [6], a classical construction of triangle-free graphs with arbitrarily large chromatic number, due to Zykov [18], shows that the constant c in the first question must be at least 4. This can also be deduced from the graph represented in Figure 1: $R(3, 5)$ does not contain a wheel as an induced subgraph, but has no 3-coloring because it has 13 vertices and stability number 4. The above questions seem difficult and it might be of interest to study subclasses of graphs with no wheels as induced subgraphs. Two such classes have already been studied (but not motivated by the study of wheel-free graphs). First, note that any wheel different from K_4 contains a cycle with a unique chord. So, graphs with no wheels as induced subgraphs is a superclass of graphs with no cycles with a unique chord and no K_4. These graphs have a precise structural description, see [14].

A natural subclass of graphs with no wheel as an induced subgraph is the class of graphs that do not contain a subdivision of a wheel as an induced subgraph. It is easy to see that this class of graphs is precisely the class of graphs with no wheel and no subdivision of K_4 as a induced subgraphs. It turns out that the structure of a graph G with no induced subdivision of K_4

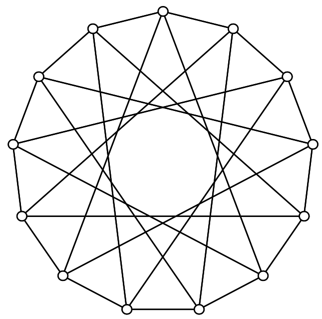

Figure 1: The Ramsey graph $R(3,5)$, that is the unique graph G satisfying $|V(G)| \geq 13$, $\alpha(G) = 4$ and $\omega(G) = 2$.

is investigated in [11]. The proof goes through several cases: when G contains $K_{3,3}$, when G contains a *prism* (another Truemper's configuration, not worth defining here), and when G contains a wheel. This last case seems to be the most difficult: no satisfactory structural description is found in this case, whereas when excluding induced wheels, a very precise structure theorem is given, with several consequences. For instance, it is proved that every graph that does not contain a subdivision of K_4 or a wheel (as induced subgraphs) is 3-colorable.

Here we restrict our attention to another subclass: graphs with no wheels *as subgraphs*. So, from here on, *contain* and *-free* refer to the subgraph containement relation.

In Section 2 we show several examples of wheel-free graphs. This will give insight to the reader and also shows that the results presented here apply to a class of graphs that is not empty. In Section 3, we prove several technical lemmas that are all slight variations on Menger's Theorem needed in the rest of the paper. In Section 4, we study the connectivity of wheel-free graphs. The main result here is that every 3-connected wheel-free graph is in fact minimally 3-connected. As a direct application, we prove that any wheel-free graph has a vertex of degree at most 3. This is a particular case of the following theorem due to Turner.

Theorem 1 (Turner [16]). *Let $k \geq 3$ be an integer and G be a graph that does not contain a cycle together with a vertex that has at least k neighbors in the cycle. Then, G has at least one vertex of degree at most k.*

Note that the result stated in [16] is slightly weaker than Theorem 1, but the proof given by Turner in [16] exactly proves the version given here. We still include our proof that wheel-free graphs have vertices of degree at most 3 in Section 4. It is not as direct as Turner's, but it illustrates some technics that we use later. Also, with the same method, we prove that every planar wheel-free graph has a vertex of degree at most 2.

In a wheel-free graph, any vertex v with three neighbors x, y, z is such that deleting v results in a graph where no cycle goes through x, y, z. In Section 5, we recall a theorem due to Watkins and Mesner [17], that describes the structure of a graph where no cycle goes through three given vertices x, y, z. We give a new shorter proof of this theorem. In Section 6 we give an application of it: we prove that any 3-connected wheel-free graph contains a pair of vertices that are not adjacent and have exactly the same neighborhood. In fact, we need to prove slightly more: the outcome holds not only for wheel-free graphs, but also for a slightly larger class of graphs: *almost wheel-free graphs* (to be defined later). This result is then used in Section 7 to show the following.

Theorem 2. *Every wheel-free graph contains either a vertex of degree at most 2 or a pair of non-adjacent vertices of degree 3 that have the same neighborhood.*

The main result of the paper follows easily.

Corollary 3. *Every wheel-free graph is 3-colorable.*

Proof. By induction on the number of vertices of a wheel-free graph G. If $|V(G)| = 1$, then G is 3-colorable. Otherwise, by Theorem 2, either G contains a vertex w of degree at most 2, or a pair $\{u, v\}$ of non-adjacent vertices with the same neighborhood. In the first case, we color $G - w$ by the induction hypothesis, and give to w one of the three colors not used in its neighborhood. In the second case, we color $G - u$ by the induction hypothesis, and give to u the same color as v. □

In several papers about Truemper's configurations, rims of wheels are required to be of length at least 4, i.e. K_4 does not count as a wheel. In Section 8, we show that this requirement does not matter much for what we are doing here: any graph that does not contain a wheel with a rim of length at least 4 is 4-colorable.

2 A wheel-free zoo

Wheel-free graphs with quite arbitrary shapes can be obtained by taking any graph, and subdividing edges until every vertex of degree at least 3 has all its neighbors (except possibly 2) of degree at most 2. Indeed in a graph obtained that way, no vertex can be the center of a wheel. But those graphs are not 3-connected (they have vertices of degree 2). In Figure 2, several 3-connected wheel-free graphs are represented. They all have similar shapes, but from that similarity, we could not deduce any general construction for all 3-connected wheel-free graphs.

Note that all graphs from Figure 2 are bipartite. However, the graph represented in Figure 3 on the left has a cycle on 15 vertices, while being 3-connected and wheel-free. On the right is represented another wheel-free graph, with a seemingly different shape.

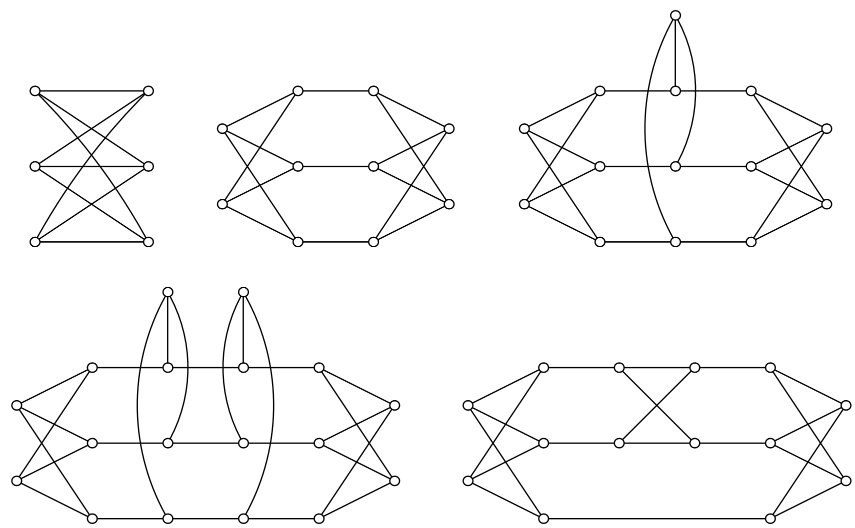

Figure 2: Some 3-connected wheel-free graphs.

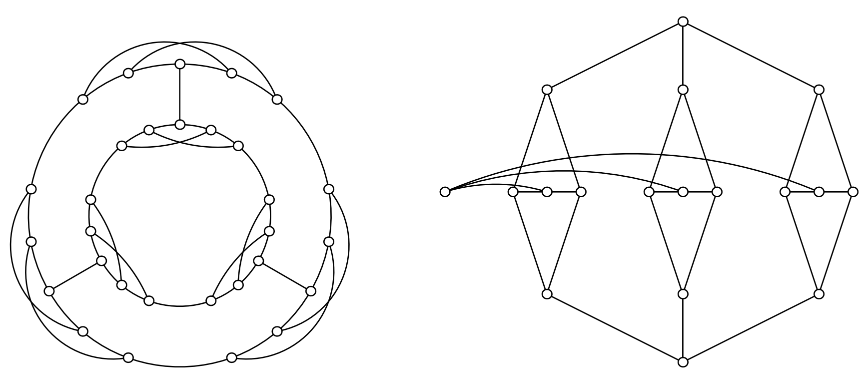

Figure 3: A 3-connected wheel-free graph with a cycle on 15 vertices (on the left). On the right, another wheel-free 3-connected graph.

From all the graphs represented so far and in view of Theorem 2, it might be asked whether wheel-free graphs that are subdivisions of 3-connected graphs are the graphs obtained from a diamond (i.e. the graph obtained from the complete graph on four vertices by removing an edge) by randomly duplicating vertices of degree 3 with neighbors of degree 2, and subdividing edges. The graph represented on the left in Figure 4 is a counter-example: it is a wheel-free subdivision of a 3-connected graph, but it cannot be obtained that way. The graph represented on the right is 2-connected in quite a strong sense: none of its subgraphs is 3-connected. However, its minimum degree is 3.

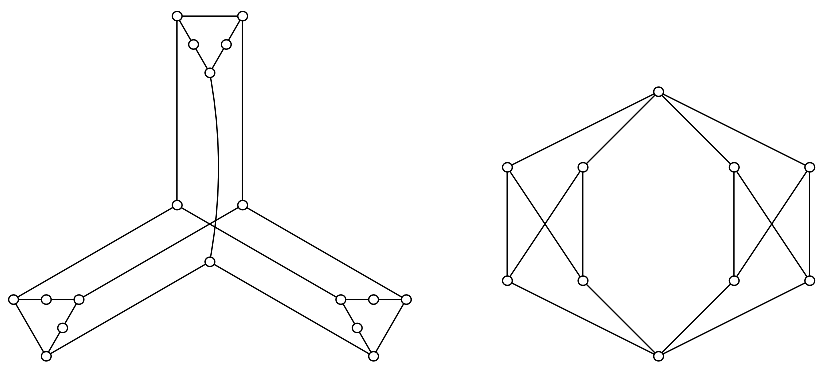

Figure 4: Wheel-free graphs with connectivity 2

3 Variations on Menger's Theorem

In this section, we present several lemmas that we will need later. They all follow very easily from Menger's Theorem. We refer to [2] for the statement of this theorem.

Paths of length 0 are allowed (they are made of one vertex). We use the following standard notation: when G is a graph and X a subset of its vertices, we denote by $G - X$ the graph obtained from G by deleting vertices from X. When $G - X$ is disconnected, we say that X is a *cutset*. When v is a vertex, we sometimes write $G - v$ instead of $G - \{v\}$. A *cutvertex* of a graph G is a vertex such that $G - v$ is disconnected. When e is an edge of G, we denote by $G \setminus e$ the graph obtained from G by deleting e (note that the ends of e are vertices of $G \setminus e$).

In some situation we need kinds of separations where some sets are allowed to intersect, so we need to define them precisely. Let $\{a, b\}$, $\{c, d\}$ be two sets of vertices of a graph G such that $a \neq b$ and $c \neq d$. Note that the two sets may intersect or be equal. A vertex v is an $(\{a, b\}, \{c, d\})$-*separator* if in $G - v$, there is no path with one end in $\{a, b\}$ and the other end in $\{c, d\}$. Note that v

can be one of a, b, c, d. The following is a rephrasing of Menger's Theorem in a particular case.

Lemma 4. *Let G be a graph and $\{a, b\}$, $\{c, d\}$ two sets of vertices of G such that $a \neq b$ and $c \neq d$. Either there exists an $(\{a, b\}, \{c, d\})$-separator or there exists two vertex-disjoint paths from $\{a, b\}$ to $\{c, d\}$.*

Note that the statement above is true when $\{a, b\} = \{c, d\}$. In this case, the two paths are of length 0.

Let $k \geq 1$ be an integer, G a graph, $Y \subseteq V(G)$ a set on at least k vertices, and $x \notin Y$ a vertex of G. A family of k paths from x to Y whose only common vertex is x and whose internal vertices are not in Y, is called a *k-fan from x to Y*. The next two results are classical (see [2]).

Lemma 5 (Fan Lemma)**.** *Let G be a k-connected graph, x a vertex of G and Y a subset of $V(G) \setminus \{x\}$ of cardinality at least k. Then there is a k-fan from x to Y.*

Theorem 6 (Dirac [5], see also [2])**.** *Let S be a set of k vertices of a k-connected graph G where $k \geq 2$. Then there is a cycle of G that contains all the vertices of S.*

Let C be a cycle of a graph G and v a vertex not in C. We say that a set on at most two vertices $\{a, b\}$ is a *(v, C)-separator* if $v \notin \{a, b\}$ and if $G - \{a, b\}$ contains no path from v to $V(C) \setminus \{a, b\}$. Note that $\{a, b\}$ and C may intersect. The following is another rephrasing of Menger's Theorem in a particular case.

Lemma 7. *In a graph G, if C is a cycle and v is a vertex not in C such that there exists no (v, C)-separator, then G contains a 3-fan from v to C.*

Lemma 8. *Let G be a graph, C a cycle of G and x a vertex not in C such that there exists no $(x,\ C)$-separator. Let y, z be two distinct vertices of C. Then there exists a cycle of G that goes through x, y and z.*

Proof. Cycle C is edge-wise partitioned into two paths $Q = y \dots z$ and $R = y \dots z$. By Lemma 7, there exists a 3-fan made of $P_1 = x \dots c_1$, $P_2 = x \dots c_2$ and $P_3 = x \dots c_3$, from x to C. From the pigeon-hole principle, at least two vertices of $\{c_1, c_2, c_3\}$ are in Q or in R, say $c_1, c_2 \in Q$. Suppose up to a relabelling that y, c_1, c_2, z appear in this order along Q. Then $yQc_1P_1xP_2c_2QzRy$ is a cycle that goes through x, y, z. □

The following is the basic tool to characterize the situation when no cycle goes through three given vertices of a 2-connected graph. Note that contrary to Theorem 19, it is not an "if and only statement".

Lemma 9. *Let G be a 2-connected graph and x, y, z be three vertices of G. Then either*

- *a cycle of G goes through x, y, z; or*

- *x, y, z are distinct and there exist two distinct vertices $t_A, t_B \notin \{x, y, z\}$ and six internally vertex-disjoint paths $P_A = t_A \dots x$, $P_B = t_B \dots x$, $Q_A = t_A \dots y$, $Q_B = t_B \dots y$, $R_A = t_A \dots z$ and $R_B = t_B \dots z$.*

Proof. Since G is 2-connected, we know that x, y and z are distinct (or a cycle goes through them) and there exists a cycle C that goes through x, z. Cycle C is edge-wise partitioned into two paths S_A and S_B from x to z. Since G is 2-connected, if $y \notin V(C)$, then there exists a 2-fan from y to C, formed by $Q_A = y \dots t_A$ and $Q_B = y \dots t_B$ say. If $t_A, t_B \in V(S_A)$, then up to symmetry, x, t_A, t_B, y appear in this order along S_A and $xS_At_AQ_AyQ_Bt_BS_AzS_Bx$ is a cycle through x, y, z. Similarly, if $t_A, t_B \in V(S_B)$, then one finds such a cycle. Hence, we may assume $t_A \in V(S_A) \setminus \{x, z\}$ and $t_B \in V(S_B) \setminus \{x, z\}$. We let $P_A = xS_At_A$, $R_A = zS_At_A$, $P_B = xS_Bt_B$ and $R_B = zS_Bt_B$. □

4 Connectivity of wheel-free graphs

The connectivity of a graph G is denoted by $\kappa(G)$. An edge e of a graph G is *essential* if $\kappa(G \setminus e) < \kappa(G)$. A graph with connectivity k and such that all its edges are essential is *minimally k-connected*. Our goal in this section is to prove that every 3-connected wheel-free graph is minimally 3-connected. This will be of use because of the following well known theorems.

Theorem 10 (Mader [12], see also [1])**.** *If G is a minimally 3-connected graph, then every cycle of G contains a vertex of degree 3.*

Theorem 11 (Mader [12], see also [1])**.** *If G is a minimally 3-connected graph, then G has at least $\frac{2|V(G)|+2}{5}$ vertices of degree 3.*

For every graph G, we denote by $W(G)$ the set of all vertices u of G such that at least one wheel of G is centered at u.

Lemma 12. *If G is a 4-connected graph, then $W(G) = V(G)$. In particular a wheel-free graph has connectivity at most 3.*

Proof. If G is 4-connected, then any vertex v has at least four neighbors. Since $G - v$ is 3-connected, by Theorem 6, it contains a cycle going through three neighbors of v. Together with v, this cycle forms a wheel centered at v. □

If $A \subset V(G)$, we denote by $N(A)$ the set of vertices from $V(G) \setminus A$ adjacent to at least one vertex of A. When $F \subseteq V(G)$, we denote by $\overline{F}$ the set $V(G) \setminus (F \cup N(F))$. We say that F is a *fragment* of G if $|N(F)| = \kappa(G)$ and $\overline{F} \neq \emptyset$ (note that if F is a fragment of G, then so is $\overline{F}$). An *end* of G is a fragment not containing other fragments as proper subsets. It is clear that any fragment F contain an end, and that consequently all graphs contain at least two disjoints ends: one in F, another one in $\overline{F}$.

Lemma 13. *Let G be a wheel-free graph such that $\kappa(G) = 2$ and F be an end of G such that $|F| \geq 2$ and $N(F) = \{a, b\}$. Let G_F be the graph obtained from*

$G[F \cup \{a,b\}]$ by adding the edge ab (if it is not there already). Then G_F is 3-connected and $W(G_F) \subseteq \{a,b\}$.

Proof. Note that $|V(G_F)| \geq 4$. Let us suppose by way of contradiction that G_F admits a cutset of cardinality 2, say $\{u,v\}$. The set $\{a,b\}$ is clearly not a cutset of G_F, so $|\{u,v\} \cap \{a,b\}| < 2$. If $|\{u,v\} \cap \{a,b\}| = 1$, then $\{u,v\}$ is also a cutset of G which has a fragment strictly included in F, a contradiction. In the same way, if $\{u,v\} \cap \{a,b\} = \emptyset$, then, since $ab \in E(G_F)$, a and b are in the same component of $G_F \setminus \{u,v\}$. Hence any component of $G_F - \{u,v\}$ not containing a and b, is also a component of $G - \{u,v\}$ and thus a fragment strictly included in F, a contradiction. So, G_F does not contain a cutset of cardinality 2 and, as $|V(G_F)| \geq 4$, G_F is 3-connected.

Suppose that G_F contains a wheel (C,w). Since G is wheel-free, the edge ab must be an edge of that wheel, and $ab \notin E(G)$. If ab is an edge of C, then a wheel of G is obtained by replacing ab with a path from a to b with internal vertices in $\overline{F}$, a contradiction. Hence, ab is an edge incident to the center of (C,w), so $w \in \{a,b\}$. This proves $W(G_F) \subseteq \{a,b\}$. □

Lemma 14. *If a 3-connected graph G contains an edge $e = ab$ that is not essential, then $\{a,b\} \subseteq W(G)$.*

Proof. Since $G \setminus ab$ is 3-connected, there exist three vertex-disjoint paths $T_1 = aa_1 \dots b$, $T_2 = aa_2 \dots b$ and $T_3 = aa_3 \dots b$ in $G \setminus ab$.

In $G - a$, which is 2-connected, we may assume that no cycle goes through a_1, a_2 and a_3 (otherwise $a \in W(G)$). So, by Lemma 9 applied to $G - a$, there exist two vertices u, v and six internally vertex-disjoint paths $P_1 = a_1 \dots u$, $P_2 = a_2 \dots u$, $P_3 = a_3 \dots u$, $Q_1 = a_1 \dots v$, $Q_2 = a_2 \dots v$ and $Q_3 = a_3 \dots v$. We set $X = P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3$.

Because of T_1, T_2, T_3, either $b \in X$, in which case we suppose $b \in P_1$, or there exists a 3-fan from b to X in $G-a$. When $b \notin X$, from the pigeon-hole principle, at least two paths from this 3-fan end in $P_1 \cup P_2 \cup P_3$ or in $Q_1 \cup Q_2 \cup Q_3$. So, up to symmetry, if $b \notin X$, then we may assume that there exists a 2-fan from b to $P_1 \cup P_2$. It follows that (wherever b) there is a cycle in $G - a$ that goes through a_1, a_2 and b. Together with a, this cycle forms a wheel centered at a. This proves $a \in W(G)$, and $b \in W(G)$ can be proved similarly. □

A graph is *almost wheel-free* if $W(G)$ is either empty, or made of a single vertex of degree 3, or made of two adjacent vertices, both of degree 3 (this notion will be used more in the next sections). By definition, every wheel-free graph is almost wheel-free.

Corollary 15. *If G is a 3-connected almost wheel-free graph, then G is minimally 3-connected.*

Proof. Since G is 3-connected, by Lemma 12, G has connectivity 3. Let $e = uv$ be an edge of G. Suppose for a contradiction that e is not essential. Then $\deg(u), \deg(v) \geq 4$, and, by Lemma 14, $u, v \in W(G)$. This contradicts the fact that G is almost wheel-free. Hence, all edges of G are essential and so G is minimally 3-connected. □

It is tempting to use Corollary 15 to give a direct proof of the next theorem. Indeed, consider the following class C of graphs: graphs such that any subgraph has connectivity at most 2 or is minimally 3-connected. By Lemma 12 and Corollary 15, any wheel-free graph is in C. Since C is made of minimally 3-connected graphs, which have vertices of degree 3 by Theorem 11, and of graphs that are even less connected, it could be that any graph in C has a vertex of degree at most 3. But unfortunately, there exist graphs in C of minimum degree 4 (they contain wheels), see Figure 5. Note also that the next theorem is best possible in some sense, since many wheel-free graphs have no vertex of degree less than 3, as shown by the graphs represented on Figures 2 and 3.

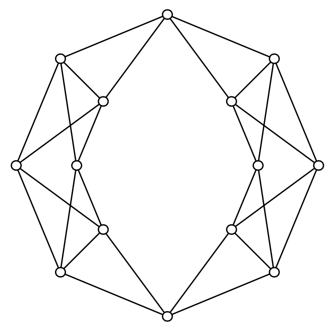

Figure 5: A graph with minimum degree 4 and no 3-connected subgraph.

Theorem 16. *If G is a wheel-free graph on at least two vertices, then G has at least two vertices of degree at most 3.*

Proof. Our proof is by induction on $|V(G)|$, the result holding trivially when $|V(G)| \leq 4$.

If G is not connected, then by the induction hypothesis, each of its components has at least one vertex of degree at most 3, so G contains at least two vertices of degree at most 3.

If G has a cutvertex a, then let C_1 and C_2 be components of $G - a$. By the induction hypothesis, $G[C_1 \cup \{a\}]$ and $G[C_2 \cup \{a\}]$ have each two vertices of degree at most 3. Thus at least one of them is distinct from a and thus is also a vertex of degree at most 3 in G. Hence, C_1 and C_2 have each at least one vertex of degree at most 3 in G.

If G is 3-connected, then, by Corollary 15, it is minimally 3-connected, so by Theorem 11, it has at least two vertices of degree at most 3.

Assume finally that G has connectivity 2. Let F and F' be two disjoints ends of G. It is enough to prove that each of F, F' contains at least one vertex of G of degree 3. Let us prove it for F, the proof being similar for F'.

If $|F| \leq 2$, this is easy to check. So, suppose $|F| \geq 3$. Let $\{u, v\} = N(F)$ and G_F be the graph as in Lemma 13. Hence G_F is 3-connected. Moreover every edge $e \neq uv$ of H is essential. Indeed if an edge different from uv were not essential, then by Lemma 14 some vertex $a \notin \{u, v\}$ would be the center of some wheel of G_F, a contradiction to Lemma 13.

Note that $G_F \setminus uv$ is a subgraph of G and so is wheel-free. Assume first that $G_F \setminus uv$ is 3-connected. Then by Corollary 15, it is minimally 3-connected. So, by Theorem 11, $G_F \setminus uv$ contains at least three vertices of degree at most 3. One of those is distinct from u and v and thus has degree at most 3 in G. Assume finally that $G_F \setminus uv$ is not 3-connected. This means that uv is essential in G_F, so, all edges of G_F are essential, so G_F is minimally 3-connected (note that G_F may contain wheels). We conclude as above by using Theorem 11 in G_F. □

With slight modifications in the proof, we shall now prove that any wheel-free planar graph on at least two vertices contains at least two vertices of degree at most 2. In fact, the key property that we use is that a planar graph does not contain a subdivision of $K_{3,3}$.

Lemma 17. *If G is a 3-connected graph that contains no subdivision of $K_{3,3}$, then $W(G) = V(G)$.*

Proof. Let v be a vertex of G. It has at least three neighbors x, y, z. If no cycle goes through them, then let $P_A, Q_A, R_A, P_B, Q_B, R_B$ be the six paths of $G - v$ (which is 2-connected) whose existence is proved in Lemma 9. Together with v, they form a subdivision of $K_{3,3}$, a contradiction. Hence a cycle C goes through x, y, z, so (C, v) is a wheel centered at v. □

Theorem 18. *If G is a wheel-free graph on at least two vertices that contains no subdivision of $K_{3,3}$, then G has at least two vertices of degree at most 2.*

Proof. The proof is very similar to the proof of Theorem 16. We start with a graph G on at least two vertices. As in the proof of Theorem 16, we may assume that G is 2-connected. So, by Lemma 17, G has connectivity 2. We consider two disjoint ends F and F' of G. It is enough to prove that both of them have cardinality 1. So, suppose for a contradiction that F has cardinality at least 2. Let $\{u, v\} = N(F)$, and G_F be the graph as in Lemma 13. So G_F is 3-connected. In addition, it contains no subdivision of $K_{3,3}$. Indeed if a subgraph H of G_F is a subdivision of $K_{3,3}$, then H contains the edge uv. So replacing uv by some path from u to v with internal vertices in $\overline{F}$ yields a subdivision of $K_{3,3}$ in G, a contradiction. Hence, by Lemma 17, any vertex of G_F is the center of a wheel. In particular, G_F contains a wheel whose center is not among u, v, a contradiction to Lemma 13. □

5 Three vertices in a cycle

The problem of deciding whether a cycle exists through three given vertices of a graph is solved from an algorithmic point of view. There is a linear time algorithm by LaPaugh and Rivest [10]. A simpler algorithm is given by Fleischner and Woeginger [7]. They also give a certificate when the answer is no, but it relies on the decomposition tree of a graph into its triconnected components,

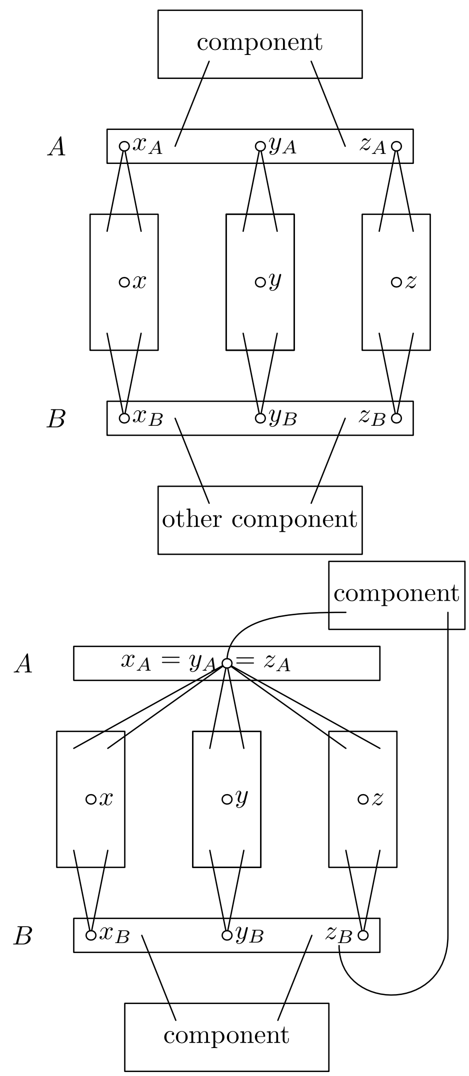

Figure 6: Two graphs with a splitter with respect to x, y and z.

see [9]. What we need here is a certificate given in terms of cutsets. The aim of this section is to state such a certificate, whose existence is proved by Watkins and Mesner [17] (see also [8] for a survey about problems of cycles through prescribed elements of a graph). We state this result in a different way (for a more convenient use in the next section), but the equivalence between the two versions is immediate. Also, we give a new proof, which is slightly shorter and, we believe, simpler.

Let G be a graph and x, y, z three distinct vertices. A pair (A, B) of two disjoint non-empty sets of vertices is a *splitter with respect to x, y, z* if (see Figure 6):

(i) x, y, z are respectively in three distinct components X, Y, Z of $G-(A\cup B)$.

(ii) All edges between X and A (resp. Y and A, Z and A) are incident to a unique vertex x_A (resp. y_A, z_A) of A.

(iii) All edges between X and B (resp. Y and B, Z and B) are incident to a unique vertex x_B (resp. y_B, z_B) of B.

(iv) $A = \{x_A, y_A, z_A\}$, $B = \{x_B, y_B, z_B\}$.

(v) Either $|A| = 1$ or $|A| = 3$. Either $|B| = 1$ or $|B| = 3$.

(vi) $G - X$, $G - Y$ and $G - Z$ are 2-connected.

(vii) If $|A| = 3$ and $|B| = 3$, then every edge between A and B is one of x_Ax_B, y_Ay_B or z_Az_B, and every component D of $G - (A \cup B)$ is such that $N(D)$ is included in either A, B, $\{x_A, x_B\}$, $\{y_A, y_B\}$ or $\{z_A, z_B\}$.

Theorem 19 (Watkins and Mesner [17])**.** *Let G be a 2-connected graph and x, y, z three vertices of G. No cycle goes through x, y, z if and only if G admits a splitter with respect to x, y, z.*

Proof. If G has a splitter it is a routine matter to check that no cycle exists through x, y, z.

Conversely, suppose that no cycle goes through x, y, z. We apply Lemma 9 to G and x, y, z: this defines six paths P_A, P_B, Q_A, Q_B, R_A and R_B. There must exist a pair $\{x_A, x_B\}$ that is an $(x,\ yQ_At_AR_AzR_Bt_BQ_By)$-separator, for otherwise, by Lemma 8, there is a cycle through x, y, z. Because of the paths P_A and P_B, we must have, up to a relabelling, $x_A \in P_A - x$ and $x_B \in P_B - x$. Let X be the component of x in $G - \{x_A, x_B\}$. We choose x_A and x_B so as to maximize the size of X. Similarly, there exists a $(y, xP_At_AR_AzR_Bt_BP_Bx)$-separator $\{y_A, y_B\}$, where $y_A \in Q_A - y$ and $y_B \in Q_B - y$. We choose y_A and y_B so as to maximize the size of the component Y of y in $G - \{y_A, y_B\}$. Finally, there exists a $(z, xP_At_AQ_AyQ_Bt_BP_Bx)$-separator $\{z_A, z_B\}$ where $z_A \in R_A - z$ and $z_B \in R_B - z$. We choose z_A and z_B so as to maximize the size of the component Z of z in $G - \{z_A, z_B\}$. Set $P = x_AP_AxP_Bx_B$, $Q = y_AQ_AyQ_By_B$ and $R = z_AR_AzR_Bz_B$.

Set $A = \{x_A, y_A, z_A\}$ and $B = \{x_B, y_B, z_B\}$. Our goal is now to prove that (A, B) is a splitter with respect to x, y, z. Conditions (i) to (iv) are satisfied from the definition of $x_A, \dots, z_B$.

Let us prove (v). Suppose $|A| = 2$. Hence, up to symmetry, we may assume that $x_A = y_A$ and $z_A \neq x_A$. We see that $x_B \neq y_B$ for otherwise, $\{x_A, x_B\}$ is a $(z, xP_At_AQ_AyQ_Bt_BP_Bx)$-separator that contradicts the maximality of Z. If in $G-(Z \cup \{x_A\})$ there exists a $(\{z_A, z_B\}, \{x_B, y_B\})$-separator u, then $u \in V(R_B)$ and $\{x_A, u\}$ is a $(z, xP_At_AQ_AyQ_Bt_BP_Bx)$-separator in G that contradicts the maximality of Z. Hence, by Lemma 4, there exists two vertex-disjoint paths in $G-(Z \cup \{x_A\})$ from $\{z_A, z_B\}$ to $\{x_B, y_B\}$. Together with x_APx_B, x_AQy_B and z_ARz_B, they form a cycle through x, y, z, a contradiction. This proves $|A| = 1$ or $|A| = 3$. Similarly, we can prove that $|B| = 1$ or $|B| = 3$.

Let us now prove (vi). Suppose for a contradiction that $G-X$ has a cutvertex u. If u is not in $P_A - x_A$ or in $P_B - x_B$, then u is a cutvertex of G, a contradiction to the 2-connectivity of G. So, up to symmetry, $u \in P_A - x_A$. Thus $\{x_B, u\}$ is an $(x,\ yQ_At_AR_AzR_Bt_BQ_By)$-separator, a contradiction to the maximality of X. Hence $G - X$ is 2-connected. Similarly, $G - Y$ and $G - Z$ are 2-connected. This proves (vi).

Let us now show an intermediate statement.

(1) If $|A| = 3$ and $|B| = 3$, then there exist two subgraphs G_A and G_B of $G - (X \cup Y \cup Z)$ such that:

(a) G_A and G_B are vertex-disjoint;

(b) G_A contains x_A, y_A, z_A and G_B contains x_B, y_B, z_B;

(c) G_A and G_B are 2-connected.

For any graph H we define the parameter $c(H) = \sum_{v \in V(H)} (\mathrm{comp}(H - v) - 1)$ where $\mathrm{comp}(H - v)$ denotes the number of components of $H - v$.

Let (G_A, G_B) be a pair of connected graphs that satisfy (a) and (b) and such that $c(G_A) + c(G_B)$ is minimum. We refer to this property as the *minimality of* (G_A, G_B). Note that such a pair (G_A, G_B) exists because the two graphs $(x_AP_At_A) \cup (y_AQ_At_A) \cup (z_AR_At_A)$ and $(x_BP_Bt_B) \cup (y_BQ_Bt_B) \cup (z_BR_Bt_B)$ are connected and satisfy (a) and (b).

Let us prove (c) by contradiction. Therefore suppose that one of G_A and G_B, say G_A, has a cutvertex v_A. If x_A, y_A, z_A are all in the same graph $G_A[C \cup \{v_A\}]$ where C is a component of $G_A - v_A$, then $(G_A[C \cup \{v_A\}], G_B)$ contradicts the minimality of (G_A, G_B). So, without loss of generality, we may assume that x_A is in a component C_A of $G_A - v_A$ and that y_A, z_A are not in C_A. We suppose moreover that v_A is chosen so as to maximize the size of C_A. If G_B admits a vertex v_B such that x_B is in a component C_B of $G_B - v_B$ and y_B, z_B are not in C_B, then we choose v_B such that the component C_B of $G_B - v_B$ that contains x_B is maximal. Else, we set $v_B = x_B$ and $C_B = \emptyset$.

In G, $\{v_A, v_B\}$ is not an $(x,\ yQ_At_AR_AzR_Bt_BQ_By)$-separator because of the maximality of X. So, there exists a path S of G with one end s in $C_A \cup C_B$, the other end s' in $(G_A \cup G_B) - (\{v_A, v_B\} \cup C_A \cup C_B)$, no internal vertex of which is in $V(G_A) \cup V(G_B) \cup X \cup Y \cup Z$ and no edge of which is in $E(G_A) \cup E(G_B)$. Up to symmetry, we assume $s \in C_A$. We have $s' \in G_B - (\{v_B\} \cup C_B)$, for otherwise $(G_A \cup S, G_B)$ contradicts the minimality of (G_A, G_B) because $\text{comp}(G_A - v_A) > \text{comp}(G_A \cup S - v_A)$ and for every internal vertex t of S, the graph $G_A \cup S - t$ is connected.

If G_B admits an $(\{x_B, s'\}, \{y_B, z_B\})$-separator w, then w is such that x_B and v_B are in a component C of $G_B - w$ and y_B, z_B are not in C. So w contradicts the maximality of C_B. Thus in G_B no $(\{x_B, s'\}, \{y_B, z_B\})$-separator exists. Hence, by Lemma 4, in G_B, up to symmetry between y_B and z_B, there are two vertex-disjoint paths $T_B = x_B \dots y_B$ and $T'_B = s' \dots z_B$. In C_A, there is a path T_A from s to x_A. In $G_A - C_A$ there is a path T'_A from y_A to z_A (because $G_A - C_A$ is connected since G_A is connected and C_A is a component of $G_A - v_A$). We observe that $P \cup Q \cup R \cup S \cup T_A \cup T'_A \cup T_B \cup T'_B$ is a cycle that goes through x, y, z, a contradiction. This proves (1).

To finish the proof, suppose for a contradiction that Conditions (vii) fails. This means whithout loss of generality that $|A| = |B| = 3$ and there exists a path S from x_A to y_B in G which contains no vertex of $\{y_A, z_A, x_B, z_B\}$. Path S has one end in G_A and one end in G_B and G_A, G_B are vertex-disjoint, so S contains a subpath S' with one end s_A in G_A, one end s_B in G_B, no internal vertex in $G_A \cup G_B$ and no edge of S' is an edge of $G_A \cup G_B$. Note that S' contains no vertex of $\{y_A, z_A, x_B, z_B\}$. We reach a contradiction by considering two cases.

Case 1: in G_A there exist two vertex-disjoint paths $T_A = x_A \dots s_A$, $T'_A = y_A \dots z_A$; or in G_B, there exist two vertex-disjoint paths $T_B = y_B \dots s_B$, $T'_B = x_B \dots z_B$. Up to symmetry, we suppose that T_A and T'_A exist. Let us apply Lemma 4 in G_B. An $(\{x_B, s_B\}, \{y_B, z_B\})$-separator would be a cutvertex of G_B, a contradiction to (1). So, there exist two vertex-disjoint paths T_B, T'_B between $\{x_B, s_B\}$ and $\{y_B, z_B\}$. Note that $\{x_B, s_B\}$ has two elements because S' has no vertex in $\{y_A, z_A, x_B, z_B\}$ (but $s_B = y_B$ is possible). We see that $S' \cup P \cup Q \cup R \cup T_A \cup T'_A \cup T_B \cup T'_B$ is a cycle through x, y, z, a contradiction.

Case 2: we are not in Case 1. We apply Lemma 4 in G_A to $\{x_A, y_A\}$ and $\{s_A, z_A\}$. Since we are not in Case 1, this gives two vertex-disjoint paths $T_A = x_A \dots z_A$ and $T'_A = y_A \dots s_A$. We apply Lemma 4 in G_B to $\{x_B, y_B\}$ and $\{s_B, z_B\}$. Since we are not in Case 1, this gives two vertex-disjoint paths $T_B = x_B \dots s_B$ and $T'_B = y_B \dots z_B$. We see that $S' \cup P \cup Q \cup R \cup T_A \cup T'_A \cup T_B \cup T'_B$ is a cycle through x, y, z, a contradiction. □

6 Twins in 3-connected almost wheel-free graphs

Our goal in this section is to prove Theorem 25. Throughout all this section, G is an almost wheel-free 3-connected graph (recall that almost wheel-free graphs are defined before Corollary 15).

Lemma 20. *G contains no triangle.*

Proof. Let u, v and w be three pairwise adjacent vertices in G. Since G is 3-connected, v has a neighbor v' distinct from u and w. In $G - v$, there is a 2-fan from v' to $\{u, w\}$, that together with v forms a wheel centered at v. Similarly, there exist wheels centered at u and w. So, $|W(G)| \geq 3$, a contradiction. □

We denote by $K_{3,3} \setminus e$ the graph obtained from $K_{3,3}$ by removing one edge.

Lemma 21. *If G has a subgraph isomorphic to $K_{3,3} \setminus e$, then G is isomorphic to $K_{3,3}$.*

Proof. Suppose that G contains 6 vertices a, b, c, x, y and z such that there are all possible edges between $\{a, b, c\}$ and $\{x, y, z\}$ except possibly ax. If there are no other vertices, then, since G is 3-connected and there is no triangle by Lemma 20, a must be adjacent to x. So, G is isomorphic to $K_{3,3}$.

(2) In $G - \{a, b, c\}$ there is no path from x to y and no path from x to z; in $G - \{x, y, z\}$ there is no path from a to b and no path from a to c.

If P is a path of $G - \{a, b, c\}$ from x to y, then $(xPyazbx, c)$ and $(xPyazcx, b)$ are wheels. Hence $\{b, c\} \subseteq W(G)$, a contradiction because by Lemma 20, $bc \notin E(G)$. The other cases are symmetric. This proves (2).

If G has more than 6 vertices, then without loss of generality, x has a neighbor $v \notin \{a, b, c\}$. Let P, Q be a 2-fan from v to $\{a, b, c, y, z\}$ in $G - x$. If one of P, Q is from v to y or z, then, together with x, it forms a path that contradicts (2). So, P, Q is in fact a 2-fan from v to $\{a, b, c\}$. If one of P and Q ends in a, then $P \cup Q$ is a path that contradicts (2). So, P, Q is in fact a 2-fan from v to $\{b, c\}$. Without loss of generality P ends in b and Q in c. Then $(vPbyazcQv, x)$ is a wheel, so $x \in W(G)$. Symmetrically, if a has a neighbor $u \notin \{x, y, z\}$, then $a \in W(G)$. Hence, either a has a neighbor $u \notin \{x, y, z\}$, so $\{a, x\} \subseteq W(G)$ and $ax \in E(G)$, or a has no neighbor $u \notin \{x, y, z\}$ and so $ax \in E(G)$ because a has degree at least 3. In both cases, $ax \in E(G)$. Hence $(vPbxazcQv, y)$ is a wheel. So, $\{x, y\} \subseteq W(G)$ which is a contradiction because $xy \notin E(G)$ by Lemma 20. □

Two vertices u and v in a graph are *twins* if they are non-adjacent, of degree 3, and $N(u) = N(v)$.

Lemma 22. *Suppose that two distinct vertices a and b of G have three distinct common neighbors x, y, z. Then a and b are twins.*

Proof. By Lemma 20, x, y and z are pairwise non-adjacent and $ab \notin E(G)$.

Suppose that a has a neighbor a' not in $\{x, y, z\}$. In $G - a$, there is a 2-fan P, Q from a' to $\{x, y, z, b\}$. We choose such a 2-fan which minimizes $|V(P) \cup V(Q)|$. If the ends of P and Q are both in $\{x, y, z\}$, then P, Q, a and b form a wheel centered at a. This is a contradiction because a has degree at least 4. Hence, we may assume up to symmetry that $P = a' \dots x$ and $Q = a' \dots b$.

(3) z has no neighbors in $\{y\} \cup V(P) \cup V(Q - b)$.

We already showed that z is not adjacent to y. If z has neighbors in $Q-b$, then there exists a 2-fan from a' to $\{x,z\}$ in $G-a$, a contradiction as above. If z has a neighbor in P, then from the minimality of P and Q, this neighbor must be the neighbor x' of x along P. But then, $\{a,b,x',x,y,z\}$ induces $K_{3,3}\setminus e$ or $K_{3,3}$, and by Lemma 21, G must be isomorphic to $K_{3,3}$, a contradiction since $|V(G)|\geq 7$ because of a'. This proves (3).

(4) If u and v are distinct vertices of $G[V(P)\cup V(Q)\cup\{a,y\}]$, then there exists a path R from u to v in $G[V(P)\cup V(Q)\cup\{a,y\}]$, that contains a and b.

The outcome is clear when u,v,a,b or v,u,a,b appear in this order in some cycle of $G[V(P)\cup V(Q)\cup\{a,y\}]$.

Suppose $u\in V(Q)$. If $v\in V(Q)\cup\{a\}$, then u,v,a,b or v,u,a,b appear in this order in $a'Qbyaa'$. So suppose $v\notin V(Q)\cup\{a\}$, and in particular $v\notin\{a',b\}$. If $v\in V(P)$, then $R=uQbyaxPv$. If $v=y$, then $R=uQbxav$. So, we may assume $u\notin V(Q)$ and symmetrically $v\notin V(Q)$.

Suppose $u\in V(P)$. Recall $u\neq a'$. If $v\in V(P)\cup\{a\}$, then u,v,a,b or v,u,a,b appear in this order in $a'Pxbyaa'$. If not, then $v=y$, and $R=uPxbQa'av$. So we may assume $u\notin V(P)$, and symmetrically $v\notin V(P)$.

Then $\{u,v\}=\{a,y\}$ and $R=axby$. This proves (4).

Since G is 3-connected and by (3), z has a neighbor $z'\notin V(P)\cup V(Q)\cup\{a,y\}$. Let $S=z'\dots u, T=z'\dots v$ be a 2-fan in $G-z$ from z' to $V(P)\cup V(Q)\cup\{a,y\}$. Let R be the path obtained in (4). Now, $S\cup T\cup R$ is a cycle in which z has at least three neighbors (namely z', a and b). Hence z is the center of some wheel of G. Similarly, the existence of a wheel centered at y can be proved. So $\{y,z\}\subseteq W(G)$, a contradiction since $yz\notin E(G)$. □

Lemma 23. *Let F be a fragment of G with $N(F)=\{a,b,c\}$. Consider the graph G_F built from $G[F\cup\{a,b,c\}]\setminus\{ab,bc,ca\}$ as follows. If a (resp. b, c) has at least two neighbors in F, then add a new vertex a' (resp. b', c') adjacent to a (resp. b, c), otherwise put $a'=a$ (resp. $b'=b$, $c'=c$). Add two new vertices d,d' and link them to a', b' and c'.*

Then G_F is an almost wheel-free 3-connected graph.

Proof. Note that since G is 3-connected, every vertex of $\{a,b,c\}$ has at least one neighbor in F. Note that in all cases, a', b' and c' have degree 3 and are pairwise non-adjacent.

Let us first prove that G_F is 3-connected. Suppose for a contradiction that G_F has a 2-cutset $\{w,w'\}$. Observe that if $x,y\in F$, then there exist three internally vertex-disjoint paths from x to y in G, and at most one of them has vertices in $V(G)\setminus(F\cup\{a,b,c\})$. This path can be rerouted through d to obtain a path of G_F. It follows that in G_F, any pair of vertices from F can be linked by three vertex-disjoint paths. Hence all the vertices from $F\setminus\{w,w'\}$ are in the same component of $G_F-\{w,w'\}$. Therefore, to get a contradiction, it is sufficient to show that any vertex of $\{a,b,c,a',b',c',d,d'\}\setminus\{w,w'\}$ can be linked by a path of $G_F-\{w,w'\}$ to some vertex of $F\setminus\{w,w'\}$. If $\{w,w'\}\subseteq F$, then at least one of a,b,c has a neighbor in $F\setminus\{w,w'\}$, because G is 3-connected.

So $G_F - \{w, w'\}$ is connected. If $w \in F$ and $w' \notin F$, then $G_F - \{w, w'\}$ is connected, unless w is the unique neighbor of a in F, $a' \neq a$ and $w = a'$ (up to a symmetry). But this contradicts the way G_F is constructed, because when a has a unique neighbor in F, then $a = a'$. When $w, w' \notin F$, one can easily see again that $G_F - \{w, w'\}$ is connected. This proves that G_F is 3-connected.

Let us now prove that $W(G_F) \subseteq W(G)$. Let $w \in W(G_F)$ and let C be the rim of a wheel centered at w. If $w = d$, then C must go through a', b' and c'. Since these vertices have degree 3, are pairwise non-adjacent and are adjacent to d', there is a contradiction because the cycle C must contain three edges incident to d'. So $w \neq d$, and similarly $w \neq d'$. If $w = a'$, then C must go through the edges $db', dc', d'b', d'c'$, a contradiction. So $w \neq a'$, and similarly $w \neq b'$ and $w \neq c'$. If $w = a$, then we know $a \neq a'$. If C is contained in $G[F]$, then $a \in W(G)$. If not, then $C \cap G[F]$ is a path P from b to c containing at least two neighbors of a. Let $x \in V(G) \setminus (F \cup \{a, b, c\})$ be a neighbor of a. In $G - a$, consider a 2-fan Q, R from x to $\{b, c\}$. Then $P \cup Q \cup R$ is a cycle of G in which a has at least three neighbors. Hence $a \in W(G)$. If $w \in F$, then a wheel of G centered at w can be obtained by replacing some path of C with both ends in $\{a, b, c\}$ with a path from $G - F$ with the same ends. Hence, $w \in W(G)$.

We proved that $W(G_F) \subseteq W(G)$. But every vertex of F has the same degree in G and G_F, and the vertices of $\{a, b, c\}$ have degree in G_F no larger than in G. Therefore, $W(G_F)$ is either empty, or made of a single vertex of degree 3, or made of two adjacent vertices, both of degree 3. In other words, G_F is almost wheel-free. □

Note that Lemma 23 is not so easy to use in a proof by induction, because the graph G_F may have more vertices and edges than G. Also $G = G_F$ is possible. A vertex is *close to a twin* if it is either a member of a pair of twins or adjacent to a member of a pair of twins.

Lemma 24. *If every vertex of degree 3 in G that is not in $W(G)$ is close to a twin, then G contains two disjoint pairs of twins.*

Proof. Since G is 3-connected, $|V(G)| \geq 4$. If $|V(G)| = 4$, then G is isomorphic to K_4, and G contains a triangle, a contradiction to Lemma 20. So, $|V(G)| \geq 5$. Since G is minimally 3-connected, by Theorem 11, it follows that G contains at least $\lceil 12/5 \rceil = 3$ vertices of degree 3. One of them, say v, is not in $W(G)$. Hence, v is close to a twin. It follows that G contains a pair of twins $\{a, b\}$. Let x, y, z be the three neighbors of a and b.

Suppose that x and y have a common neighbor c distinct from a and b. Then G contains a subgraph isomorphic to $K_{3,3} \setminus e$. Hence, by Lemma 21, G is isomorphic to $K_{3,3}$, so it contains two disjoint pairs of twins. Therefore, we may assume that x, y have no common neighbors (except a and b), and similarly, x, z and y, z. In particular, no pair of twins of G contains x, y, or z. Let $R = V(G) \setminus \{a, b, x, y, z\}$. Note that $|R| \geq 3$ because of the neighbors of x, y and z.

We claim that R contains a vertex $u \notin W(G)$ of degree 3 in G. Suppose first that $G[R \setminus W(G)]$ has no vertex of degree at most 1. Then, $G[R \setminus W(G)]$

contains a cycle C, which is also a cycle in G. By Corollary 15 G is minimally 3-connected, and so, according to Theorem 10, cycle C contains a vertex u whose degree (in G) is 3. Hence, we are left with the case when $G[R \setminus W(G)]$ has a vertex u of degree at most 1. The degree of u in G is at most 3. Indeed, u is adjacent to at most one vertex among x, y, z and to at most one vertex in $W(G)$ because $W(G)$ is a clique and by Lemma 20, there is no triangle in G. This proves our claim.

Now u is not in $W(G)$, is close to a twin, so it must be a member of a pair of twins of G, or adjacent to some member of a pair of twins of G. This pair of twins is in R, so, it is disjoint from $\{a, b\}$. □

Theorem 25. *If G is an almost wheel-free 3-connected graph, then G contains two disjoint pairs of twins.*

Proof. We consider a minimum counter-example G, and we look for a contradiction.

(5) No fragment F of G is such that $|F| \geq 6$, $|\overline{F}| \geq 2$ and F contains a pair of twins of G.

For suppose that such a fragment F exists with $N(F) = \{a, b, c\}$. So, $\overline{F}$ is also a fragment of G and $N(\overline{F}) = \{a, b, c\}$. Consider the graph $G_{\overline{F}}$ built as in Lemma 23. Since $|F| \geq 6$, we have $|V(G_{\overline{F}})| < |V(G)|$. By Lemma 23, $G_{\overline{F}}$ is almost wheel-free and 3-connected.

By the minimality of G, $G_{\overline{F}}$ contains two disjoint pairs of twins. None of them is a pair of twins in G, for otherwise with the one in F, G would have two pairs of twins, a contradiction. Hence one pair of twins is $\{d, d'\}$ and the second one is $\{a, b\}$, $\{a, c\}$ or $\{c, b\}$, because if $a' \neq a$ (resp. $b' \neq b$, $c' \neq c$), then a' (resp. b', c') is in no pair of twins. Without loss of generality $\{a, b\}$ is a pair of twins of $G_{\overline{F}}$. This means that a and b are adjacent to d, to d' and to a vertex $d'' \in \overline{F}$, and that d'' is the unique neighbor of a and b in $\overline{F}$. It follows that $\{d'', c\}$ is a cutset of $G_{\overline{F}}$, a contradiction, unless $\overline{F} = \{d''\}$, which is also a contradiction because we suppose $|\overline{F}| \geq 2$. This proves (5).

By Lemma 24, and because G does not contain two disjoint pairs of twins, there exists a vertex v in $V(G) \setminus W(G)$ of degree 3 and not close to a twin. Let x, y and z be the three neighbors of v. Note that $G - v$ is 2-connected and in $G - v$, no cycle goes through x, y, z (because such a cycle would be the rim of a wheel centered at $v \notin W(G)$). Hence, by Theorem 19, there is a splitter $A = \{x_A, y_A, z_A\}$, $B = \{x_B, y_B, z_B\}$ for x, y, z in $G - v$. We denote by X, Y, Z the components of $G - (A \cup B)$ that contain x, y, z respectively.

(6) Either $|Y| = 1$, or $|Y| \geq 4$, or $Y = \{y, y', y''\}$ and $y_A y'$, $y_A y''$, $y_B y'$, $y_B y'' \in E(G)$.

Suppose Y has cardinality 2, say $Y = \{y, y'\}$. Since y' has degree at least 3 and is non-adjacent to v, y' must be adjacent to y_A, y_B, and y. Since y also has degree at least 3, it must be adjacent to at least one of y_A, y_B. Hence, G contains a triangle, a contradiction to Lemma 20.

Suppose Y has cardinality 3, say $Y = \{y, y', y''\}$. If $yy' \notin E(G)$ then, since y' has degree at least 3, it must be adjacent to y'', y_A and y_B. Also, y'' must be adjacent to at least one of y_A or y_B, so G contains a triangle, a contradiction to Lemma 20. Hence, $yy' \in E(G)$ and similarly, $yy'' \in E(G)$. So by Lemma 20 $y'y'' \notin E(G)$. Since y' and y'' have degree at least 3, $y_Ay', y_Ay'', y_By', y_By'' \in E(G)$ as claimed. This proves (6).

(7) Either $|X| = 1$ *or* X *contains a pair of twins of* G*.*

Suppose $|X| > 1$. Note that X is a fragment of G. So let us build the graph G_X as in Lemma 23 (with more convenient names given to the vertices): start with $G[X \cup \{v, x_A, x_B\}]$ and add two vertices y' and z' and the edges vy' and vz'. If x_A has at least two neighbors in X, then add a new vertex x'_A and the edges $x_Ax'_A$, x'_Ay' and x'_Az'; otherwise, set $x'_A = x_A$ and add the edges x_Ay' and x_Az'. If x_B has at least two neighbors in X, then add a new vertex x'_B and the edges $x_Bx'_B$, x'_By' and x'_Bz'; otherwise, set $x'_B = x_B$ and add the edges x_By' and x_Bz'. By Lemma 23, G_X is almost wheel-free and 3-connected.

We claim that $|V(G_X)| < |V(G)|$. Observe that G_X has at most four vertices not in G, the ones of $\{z', y', x'_A, x'_B\}$. Suppose for a contradiction that $|V(G_X)| \geq |V(G)|$. Then $|Y| + |Z| \leq 4$. By (6), one of Y and Z, say Z, has cardinality 1. If $|Y| > 1$, then by (6), $Y = \{y, y', y''\}$ and y_Ay', y_Ay'', y_By', $y_By'' \in E(G)$. Moreover $A = \{x_A\}$ and $B = \{x_B\}$ for otherwise $|V(G_X)| < |V(G)|$. Thus x_A and x_B have three common neighbors in G, namely y', y'' and z, and also have degree at least 4. This contradicts Lemma 22. Hence $|Y| = |Z| = 1$. By Lemma 20, $y_Ay_B \notin E(G)$ and $z_Az_B \notin E(G)$. If $y_A = z_A$ and $y_B = z_B$, then y and z have three common neighbors in G, namely x_A, x_B and v. Hence, by Lemma 22, they form a pair of twins of G, so v is close to a twin, a contradiction to the choice of v. Hence by symmetry, we may assume that $|A| = 3$. Hence $|V(G_X)| < |V(G)|$, unless $|B| = 1$, $x_A \neq x'_A$, $x_B \neq x'_B$, and $V(G) = A \cup B \cup X \cup \{y, z, v\}$. Since there is no triangle, one of y_A, z_A is of degree 2, a contradiction. This proves the claim.

Now, G_X is almost wheel-free, 3-connected and smaller than G. Hence, by the minimality of G, G_X contains two disjoint pairs of twins. One of them is $\{y', z'\}$. The other one is either in X, in which case it is also a pair of twins of G (what we want to prove) or is $\{x_A, x_B\}$. In the later case, x_A and x_B have degree 3 in G_X, so they are both adjacent to a unique same vertex x' in X. Then $\{x', v\}$ is a cutset of G_X, that separates $X \setminus \{x'\}$ from x_A, a contradiction to the 3-connectivity of G_X. This proves (7).

(8) $|X| = |Y| = |Z| = 1$.

By (7) every set among X, Y or Z of cardinality at least two contains a pair of twins. Hence we may assume that $|Y| = |Z| = 1$. Thus y_Ay_B and z_Az_B are not edges by Lemma 20. Suppose $|X| > 1$. Note that by (7), X contains a pair of twins of G.

Suppose first that $|A| = |B| = 3$. Hence, since G is 3-connected, every component D of $G - (A \cup B \cup \{v\})$ satisfies $N(D) \subseteq A$ or $N(D) \subseteq B$. Let

C_A be the union of all components D such that $N(D) \subseteq A$. Because G has minimum degree 3 and no triangle, there must be at least two vertices in C_A. Hence, $F = V(G) \setminus (C_A \cup \{x_A, y_A, z_A\})$ contradicts (5).

Suppose that $|B| = 1$ and $|A| = 3$. The vertices x_A, y_A and z_A have degree at least 3, and by (5), there is at most one vertex in the union of all components D such that $N(D) \subseteq A$ in $G - (A \cup B \cup \{v\})$. If such a component D exists, then there is a common neighbour to x_A, y_A and z_A and those three vertices are pairwise non-adjacent. Hence there must be a component D' such that $x_B \in N(D')$ because of their degrees. If no component D such that $N(D) \subseteq A$ exists, then since x_A, y_A and z_A do not form a triangle, there also must be a component D' such that $x_B \in N(D')$. Let x'_B be a neighbor of x_B in D'. In $G - x_B$, consider a 2-fan from x'_B to $\{x_A, y_A, z_A\}$. If this 2-fan is formed of $P = x'_B \dots y_A$ and $Q = x'_B \dots z_A$, then $(vyy_APx'_BQz_Axv, x_B)$ is a wheel, a contradiction because x_B has degree at least 4. Hence, we may assume that the union of the two paths of this 2-fan is a path P from x_A to y_A with internal vertices in C_B and going through x'_B. Now, consider a neighbor $x' \in X$ of x_B, and a 2-fan in $G - \{x_B\}$ from x' to $\{v, x_A\}$. The union of the two paths of this 2-fan is a path Q from v to x_A, with interior in X, and that goes through x'. Then vQx_APy_Ayv is a cycle that contains three neighbors of x_B (namely y, x' and x'_B), a contradiction since x_B has degree at least 4.

Similarly, we get a contradiction if $|B| = 1$ and $|A| = 3$. So $|A| = |B| = 1$ and $\{y, z\}$ is a pair of twins, a contradiction to the choice of v. This proves (8).

(9) $|A| = |B| = 3$.

By the choice of v, $\{x, y\}$ is not a pair of twins of G, and so it is impossible to have $|A| = 1$ and $|B| = 1$. If $x_A = y_A = z_A$, then by Lemma 22, $\{v, x_A\}$ is a pair of twins of G, a contradiction to the choice of v. Similarly, $x_B = y_B = z_B$ is impossible. Hence, $|A| = |B| = 3$. This proves (9).

We are now ready to finish the proof. We know by (8) that $|X| = |Y| = |Z| = 1$ and by (9) that $|A| = |B| = 3$. Hence, every component D of $G - (A \cup B \cup \{v, x, y, z\})$ satisfies $N(D) \subseteq A$ or $N(D) \subseteq B$, and there are no edges between A and B. Let C_A (resp. C_B) be the union of all components D such that $N(D) \subseteq A$ (resp. $N(D) \subseteq B$). Then, $F = C_B \cup \{x_B, y_B, z_B, v\}$ is a fragment of G, and we build the graph G_F as in Lemma 23. Note that $N(F) = \{x, y, z\}$ and each of x, y and z has exactly one neighbor in $\overline{F}$. So, the graph G_F is obtained by adding to $G \setminus F$ two vertices d and d', and by linking them to x, y and z. We obtain a graph smaller than G, and by the minimality of G, it must contain two disjoint pairs of twins. One of them is $\{d, d'\}$, the other one must be in C_A and is in fact a pair of twins of G. Hence, C_A contains a pair of twins of G, and by a symmetric argument, so does C_B. Hence G contains two disjoint pairs of twins, a contradiction. □

7 Proof of Theorem 2

To prove Theorem 2, we actually prove the following stronger theorem.

Theorem 26. *Every wheel-free graph on at least two vertices contains either*

(i) two vertices of degree at most 2; or

(ii) one vertex of degree at most 2 and one pair of twins; or

(iii) two disjoint pairs of twins.

Proof. By induction on the number of vertices.

If $|V(G)| = 2$, then (i) obviously holds. If G is not 2-connected, then the conclusion follows easily from the induction hypothesis applied to connected components or blocks of G. If G is 3-connected, then the conclusion holds by Theorem 25. Hence, we may assume that the connectivity of G is 2. It is enough to prove that every end of G contains either a vertex of degree 2 or a pair of twins of G. So, let F be an end of G and $N(F) = \{a, b\}$. If $|F| = 1$, then F contains a vertex of degree 2, so suppose $|F| \geq 2$. Build G_F as in Lemma 13.
Case 1: $ab \in E(G)$. Then, G_F is a subgraph of G, so G_F is wheel-free and 3-connected. By Theorem 25, G_F contains 2 disjoint pairs of twins. If one of them is disjoint from $\{a, b\}$, then it is a pair of twins of G. Hence, we may assume the two disjoint pairs of twins in G_F are $\{a, a'\}$ and $\{b, b'\}$ for some $a' \neq b'$. Note that $ab', ba' \in E(G)$, so $a'b' \in E(G)$. Let a'' be the third common neighbor of b, b' and b'' be the third common neighbor of a, a'. Observe that a subgraph of G_F on $\{a, a', a'', b, b', b''\}$ is isomorphic to $K_{3,3} \setminus e$. Hence, by Lemma 21, G_F is in fact isomorphic to $K_{3,3}$. It follows that $\{a', a''\} \subseteq F$ is a pair of twins of G.

Case 2: $ab \notin E(G)$. Then $G[F \cup \{a, b\}] = G_F \setminus ab$. We first prove the following.

(10) No pair of twins of $G_F \setminus ab$ contains a (resp. b).

Otherwise, let $\{a, a'\}$ be a pair of twins in $G_F \setminus ab$. Let be the three neighbours of a in F. Since G_F is 3-connected, x has degree 3 in G_F and so has a neighbour x' in $F \setminus \{a, b\}$.

Suppose $a' = b$. A 2-fan in $G_F - x$, from x' to $\{y, z\}$ together with a, b and x form a wheel of G_F centered at x, a contradiction to Lemma 13. Hence $a' \neq b$.

In $G_F - x$, let P, Q be a 2-fan from x' to $\{y, z, b\}$. Up to symmetry, $P = x' \dots y$. If $Q = x' \dots b$, then $(x'Pya'zabQx', x)$ is a wheel in G_F, a contradiction to Lemma 13. Hence $Q = x' \dots z$. Let P', Q' be a 2-fan in $G_F - a$ from b to $\{x, y, z\}$. Note that $(V(P) \cup V(Q)) \setminus \{y, z\}$ is disjoint from $V(P') \cup V(Q')$, for otherwise, a 2-fan from x' to $\{y, b\}$ or to $\{z, b\}$ exists, giving a contradiction as above. Hence we may assume $P' = b \dots y$ and $Q' = b \dots x$ or $Q' = b \dots z$. Then $(bP'y'Px'Qzaxb, a')$ or $(bP'yPx'xazb, a')$ respectively is a wheel in G, a contradiction.

This proves (10).

If a or b has at least three neighbors in F, then we apply the induction hypothesis to $G_F \setminus ab$. If $G_F \setminus ab$ contains two vertices of degree at most 2, one of them is in F. Otherwise, F contains a pair of twins in $G_F \setminus ab$. By (10) this pair does not intersect $\{a, b\}$ and thus is also a pair of twins in G.

Hence we may assume that a and b have both exactly two neighbors in F. Thus G_F is almost wheel-free because $W(G_F) \subseteq \{a, b\}$ by Lemma 13. By Theorem 25, G_F contains two disjoint pairs of twins. If one of them is in F, it is also a pair of twins in G. If not, then the two disjoint pairs of twins in G_F are $\{a, a'\}$ and $\{b, b'\}$ for some $a' \neq b'$. As in Case 1, one shows that G_F has a subgraph isomorphic to $K_{3,3} \setminus e$, and so by Lemma 21, G_F is isomorphic to $K_{3,3}$. It follows that F contains a pair of twins of G. □

Remark that Theorem 26 is best possible as shown by the three graphs represented on Figure 7.

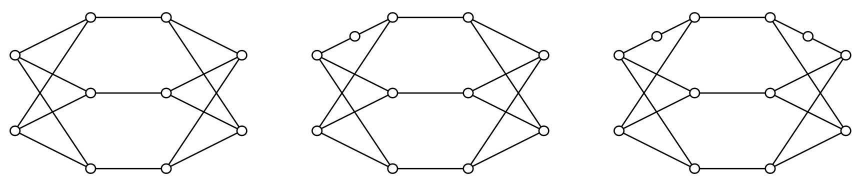

Figure 7: Three wheel-free graphs.

8 When K_4 does not count as a wheel

A *long wheel* is a wheel whose rim is a cycle of length at least 4. Observe that K_4 is the only wheel that is not a long wheel. Note that in several articles, the word *wheel* is used for what we call here *long wheel*.

Lemma 27. *Let G be a long-wheel-free graph. Then every block of G is wheel-free or isomorphic to K_4.*

Proof. Let H be a block of G. If H does not contain K_4, then it is obviously wheel-free. So, suppose that H contains a subgraph H' isomorphic to K_4. If $H \neq H'$, then let $v \in V(H) \setminus V(H')$. A long wheel in G is obtained by taking the union of H' and a 2-fan from v to H', a contradiction. □

Theorem 28. *If a graph does not contain a long wheel as a subgraph, then it is 4-colorable.*

Proof. It is enough to prove that every block H of G is 4-colorable. This is obvious if H is isomorphic to K_4. Otherwise, by Lemma 27, H is wheel-free, so it is even 3-colorable by Theorem 3. □

Note that to prove the theorem above, we do not need the full strength of Theorem 3. Knowing that wheel-free graphs are 4-colorable is enough, and follows easily from Theorem 1 or Theorem 16.

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